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Exact results for quantum chains with multisite interactions

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Abstract. Generalised spin-$\frac{1}{2}$ and Potts quantum chains with m- (n-) site interactions for the $z(x)$ component of the site variables are shown to be self-dual. Through duality, with m arbitrary, $n=2$ versions of the models are transformed into m independent Ising or Potts chains in a transverse field, allowing us to get the exact values of the critical exponents for any m.

1. Introduction

Lattice statistical models with multisite interactions may lead to a rich variety of critical behaviour. Among the two-dimensional classical models let us mention the exactly solved eight-vertex model (Baxter 1972) which may be formulated as an Ising model with two- and four-spin interactions (Wu 1971, Kadanoff and Wegner 1971) and the Baxter-Wu model (Baxter and Wu 1974, Baxter 1974) which is an Ising model with three-spin interactions on the triangular lattice. The first displays continuously varying exponents and the second belongs to the $q=3$ Potts universality according to the den Nijs (1979) conjecture. Other examples may be found in the review of Nagle and Bonner (1976). Multisite interactions are often necessary in order to describe the phase diagrams of metallic alloys (Sanchez and de Fontaine 1981) or complex magnetic structures (Nagaev 1982). They have also been introduced in an approach to the spin glass problem (Derrida 1981, Gross and Mezard 1984).

In the present work we introduce a new class of one-dimensional spin-$\frac{1}{2}$ and Potts quantum models with m- (n-) site interactions for the $z(x)$ component of the site variables generalising either the Ising model in a transverse field (ITF) (Pfeuty 1970) or the quantum Potts chain (Sólyom and Pfeuty 1981, Hamer 1981) (by analogy with the spin-$\frac{1}{2}$ model we call it the Potts model in a transverse field (PTF)). In a series of papers (Turban 1982, Penson *et al* 1982, Turban and Debierre 1982) the version with $n=1$, m arbitrary, was studied using duality transformations, perturbation expansions and renormalisation group techniques. Here we show that the $n=2$, m arbitrary version is related through a duality transformation to either the Ising model in a transverse field or the Potts model in a transverse field. This allows us to get exact results for the critical exponents. After the completion of a preliminary version of this work (Turban 1984) we learned that the connection between the anisotropic XY chain ($m=n=2$, spin-$\frac{1}{2}$) and the Ising model in a transverse field had been obtained previously through the same duality transformation in the study of the XY chain with a defect

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(Peschel and Schotte 1984) and in the study of dimerised chains through a more complicated transformation (Jullien and Fields 1978).

In § 2 we study the spin-$\frac{1}{2}$ quantum chain with multisite interactions. We show that the model is self-dual for any values of m and n in § 2.1. A perturbation expansion of the spontaneous magnetisation of the $n=2$ model in § 2.2 shows that a simple relation with the ITF may be expected. In the next section (§ 2.3) we show, through a duality transformation, that this model is equivalent to m Ising chains in a transverse field. This result allows us to get the magnetic critical exponents which are m-dependent (§ 2.5). The same steps are followed in § 3 in the study of the Potts quantum chain with multisite interactions.

2. Spin-$\frac{1}{2}$ quantum chain with multisite interactions

2.1. Self-duality

Let us consider an anisotropic spin-$\frac{1}{2}$ quantum chain with Hamiltonian

$$\mathcal{H}_{mn}(\lambda) = -\sum_{j=1}^{N}\prod_{k=0}^{m-1}\sigma_z(j+k) - \lambda\sum_{j=1}^{N}\prod_{k=0}^{n-1}\sigma_x(j+k) \tag{2.1}$$

where the σ are Pauli spin operators defined in the usual way:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

The multisite interaction involves the product of m (n) successive spin components σ_z (σ_x) on each of the N sites of a linear chain with free ends. This Hamiltonian includes two well known particular cases: the Ising chain in a transverse field when $m=1$ and $n=2$ (Pfeuty 1970) and the anisotropic XY chain (Lieb *et al* 1961, Katsura 1962) when $m=n=2$. A recently studied multisite form of the ITF (Turban 1982, Penson *et al* 1982, Igloi *et al* 1983, Maritan *et al* 1984) is recovered when $n=1$. All these models are known to be self-dual. We show now that this remains true for any values of n and m.

Let the dual spin operators be defined as follows

$$\begin{aligned} \tau_x(j) &= \prod_{k\geqslant 0}\sigma_z(j+kn)\sigma_z(j+kn+m) \\ \tau_z(j) &= \prod_{k\geqslant 0}\sigma_x(j-km)\sigma_x(j-km-n). \end{aligned} \tag{2.3}$$

They commute on different sites and satisfy the Pauli algebra on the same site since

$$\begin{aligned} &\tau_x^2(j) = \tau_z^2(j) = 1 \\ &\tau_x(j)\tau_z(j) + \tau_z(j)\tau_x(j) = 0. \end{aligned} \tag{2.4}$$

With the dual Pauli spin operators equation (2.1) may be rewritten as

$$\mathcal{H}_{mn}(\lambda) = -\lambda\sum_{j=1}^{N}\prod_{k=0}^{m-1}\tau_z(j+k) - \sum_{j=1}^{N}\prod_{k=0}^{n-1}\tau_x(j+k) = \lambda\mathcal{H}_{mn}(\lambda^{-1}) \tag{2.5}$$

so that $\mathcal{H}_{mn}$ is self-dual and as a consequence, in the thermodynamic limit ($N\to\infty$), any phase transition, when unique, must occur at $\lambda_c^2 = 1$ since any singularity in the ground-state energy at $\lambda = \lambda_c$ is transformed by duality into a singularity at $\lambda = \lambda_c^{-1}$.

2.2. Perturbation expansion of the spontaneous magnetisation when $n = 2$

We now specialise to the case $n = 2$, m arbitrary and look for the spontaneous magnetisation $\langle\sigma_x\rangle_m$ through a perturbation expansion in powers of λ^{-1} (the bracket $\langle\ldots\rangle_m$ indicates a ground-state expectation value). An external field h_x is applied in the x direction and the Hamiltonian is written in a form which is more appropriate for the λ^{-1} expansion of the ground-state energy per site $e_0(h_x)$:

$$\mathcal{H}_{m2}(\lambda)/\lambda = -\sum_{j=1}^{N} \sigma_x(j)\sigma_x(j+1) - h_x \sum_{j=1}^{N} \sigma_x(j) - \lambda^{-1} \sum_{j=1}^{N} \prod_{k=0}^{m-1} \sigma_z(j+k). \tag{2.6}$$

The first two terms constitute the unperturbed Hamiltonian and the last one is the perturbation proportional to λ^{-1}. Using standard methods (Fradkin and Susskind 1978, Kogut 1979) one obtains

$$-e_0(h_x) = 1 + h_x + \frac{1}{2}\frac{\lambda^{-2}}{2+mh_x} - \frac{2m+1}{8}\frac{\lambda^{-4}}{(2+mh_x)^3} + \frac{1}{4}\frac{\lambda^{-4}}{(2+mh_x)^2(1+mh_x)} + \frac{1}{4}\frac{\lambda^{-4}}{(2+mh_x)^2}\sum_{p=1}^{m-1}\frac{1}{2+(m-p)h_x} + O(\lambda^{-6}). \tag{2.7}$$

Making use of the Feynman-Hellmann theorem, the spontaneous magnetisation is given by

$$\langle\sigma_x\rangle_m = -\left.\frac{\partial e_0}{\partial h_x}\right|_{h_x=0} = 1 - \frac{m}{8}\lambda^{-2} + \frac{m(m-8)}{128}\lambda^{-4} + O(\lambda^{-6}) \tag{2.8}$$

a result which is consistent with a simple power law

$$\langle\sigma_x\rangle_m = (1-\lambda^{-2})^{m/8} \qquad \lambda \geqslant 1 \tag{2.9}$$

for any m. Known exact results are recovered when $m = 1$ (ITF) $\beta = \frac{1}{8}$ (Pfeuty 1970) and when $m = 2$ (XY) $\beta = \frac{1}{4}$ (McCoy 1968). This result led us to conjecture the existence of a simple relation between $\mathcal{H}_{m2}$ and the Ising model in a transverse field. This relation is established in § 3 via a duality transformation.

2.3. Decoupling of the Ising chain with multisite interactions when $n = 2$

By making use of the ITF dual variables

$$\tau_x(j) = \prod_{k\geqslant 0} \sigma_z(j+k), \qquad \tau_z(j) = \sigma_x(j-1)\sigma_x(j) \tag{2.10}$$

the Hamiltonian $\mathcal{H}_{m2}(\lambda)$ is transformed into

$$\mathcal{H}_{m2}(\lambda) = -\sum_{j=1}^{N} \tau_x(j)\tau_x(j+m) - \lambda \sum_{j=1}^{N} \tau_z(j) \tag{2.11}$$

i.e. into m non-interacting ITF. It follows that the ground-state energy per site e_0, the excitation spectrum $\omega(k)$ and the mass gap ω are independent of m and given by the ITF exact results (Pfeuty 1970, Kogut 1979)

$$-e_0 = \frac{4}{\pi}(1+\lambda)E\left(\frac{\pi}{2}, \theta\right) \qquad \theta = \frac{4\lambda}{(1+\lambda)^2}$$
$$\omega(k) = 4(1+\lambda^2+2\lambda\cos k) \qquad \omega = 2|1-\lambda| \tag{2.12}$$

where E is the elliptic integral of the second kind. When an external field is applied, the m chains become coupled by the dual of the field term.

Let us first apply an external field h_z in the z direction; it introduces a term $-h_z \sum_{j=1}^{N} \sigma_z(j)$ which, by duality, is transformed into a first-neighbour coupling between the m chains $-h_z \sum_{j=1}^{N} \tau_x(j)\tau_x(j+1)$ in the dual system. It follows that for $m \geqslant 2$, one obtains

$$\lim_{h_z \to 0} \langle \sigma_z \rangle_m = \lim_{h_z \to 0} \langle \tau_x(j)\tau_x(j+1) \rangle_{\mathrm{ITF}} = \lim_{h_z \to 0} \langle \tau_x \rangle_{\mathrm{ITF}}^2 \tag{2.13}$$

since in the limit one obtains m independent chains with the same spontaneous magnetisation which according to equation (2.11) is non-vanishing below $\lambda_c = 1$.

An external field applied in the x direction introduces a term $-h_x \sum_{j=1}^{N} \sigma_x(j)$ and also couples the dual chains since $\sigma_x(j)$ is transformed into $\prod_{k \leqslant j} \tau_z(k)$. The spontaneous magnetisation $\langle \sigma_x \rangle_m$ may then be deduced from the large R behaviour of the spin-spin correlation function $\rho_{xx}(R) = \langle \sigma_x(j)\sigma_x(j+R) \rangle_m$ in zero external field. Through duality $\rho_{xx}(R)$ is transformed into

$$\rho_{xx}(R) = \left\langle \prod_{k=1}^{R} \tau_z(j+k) \right\rangle = \prod_{\alpha=1}^{m} \left\langle \prod_{k=1}^{R_\alpha} \tau_z(j_\alpha + k) \right\rangle_\alpha \tag{2.14}$$

where the last averages $\langle \ldots \rangle_\alpha$ are ground-state expectation values taken on the m independent chains and R_α is of order R/m. The inverse duality transformation performed on each of the m independent chains leads to m Hamiltonians

$$\mathcal{H}_\alpha(\lambda) = -\sum_{j_\alpha=1}^{N/m} \mu_z(j_\alpha) - \lambda \sum_{j_\alpha=1}^{N/m} \mu_x(j_\alpha)\mu_x(j_\alpha+1) \tag{2.15}$$

and gives a product of m spin-spin correlation functions such that

$$\rho_{xx}(R) = [\rho_{xx}^{\mathrm{ITF}}(R/m)]^m. \tag{2.16}$$

When $R \to \infty$ and with $\lambda \geqslant 1$, the spin-spin correlation functions give the square of the spontaneous magnetisation and

$$\langle \sigma_x \rangle_m = [\langle \mu_x \rangle_{\mathrm{ITF}}]^m. \tag{2.17}$$

These results will be used to determine the critical exponents of the multisite model $\mathcal{H}_{m2}(\lambda)$.

2.4. Scaling properties of quantum systems

In this section we briefly collect several known results concerning the scaling properties of quantum systems near a zero temperature critical point which will be used in the following sections.

Under a change of the length scale by a factor b, the excitation energy transforms according to

$$\omega(bk, \varepsilon b^{y_\varepsilon}, hb^{y_h}) = b^z \omega(k, \varepsilon, h) \tag{2.18}$$

where ε is a temperature-like variable and y_ε the associated dimension; h is an external field with dimension y_h and k the wavevector of the excitation. From equation (2.18) one easily deduces the following relations

$$\omega(k) \sim k^z \tag{2.19}$$

$$\omega(\varepsilon) \sim \varepsilon^s = \varepsilon^{z/y_\varepsilon} \tag{2.20}$$

so that the gap exponent s is related to the correlation length exponent

$$\nu = y_\varepsilon^{-1} \tag{2.21}$$

by the scaling law

$$s = \nu z \tag{2.22}$$

where z is the dynamical exponent.

The singular part of the ground-state energy per site e_0 transforms according to

$$e_0(\varepsilon b^{y_\varepsilon}, h b^{y_h}, bq) = b^{z+d} e_0(\varepsilon, h, q) \tag{2.23}$$

since e_0 is an energy density whereas ω is a total energy difference. The q variable has been introduced in order to allow for the possibility of a modulated external field with amplitude h and wavevector q. The spontaneous magnetisation behaves like

$$m(\varepsilon) \sim \varepsilon^\beta = \varepsilon^{(z+d-y_h)/y_\varepsilon} \tag{2.24}$$

in the ordered phase and at the critical point:

$$m(h) \sim h^{1/\delta} = h^{(z+d-y_h)/y_h}. \tag{2.25}$$

The zero-field susceptibility is given by:

$$\chi(\varepsilon) \sim \varepsilon^{-\gamma} = \varepsilon^{-(2y_h - z - d)/y_\varepsilon} \tag{2.26}$$

and the staggered susceptibility diverges at the critical point like:

$$\chi(q) \sim q^{-\varphi} = q^{-2y_h + z + d}. \tag{2.27}$$

The spin-spin correlation function transforms according to

$$\rho(R/b, \varepsilon b^{y_\varepsilon}) = b^{2(d+z-y_h)} \rho(R, \varepsilon) \tag{2.28}$$

so that

$$\rho(R, \varepsilon) \sim R^{-(d+z-2+\eta)} f\left(\frac{R}{\xi}\right) = R^{-2(d+z-y_h)} f\left(\frac{R}{\varepsilon^{-1/y_\varepsilon}}\right) \tag{2.29}$$

and:

$$\xi \sim \varepsilon^{-\nu} = \varepsilon^{-1/y_\varepsilon} \tag{2.30}$$

$$\eta = d + z + 2 - 2y_h. \tag{2.31}$$

2.5. Critical exponents of the spin-$\frac{1}{2}$ quantum chain with multisite interactions when $n = 2$

The Ising model in a transverse field has been exactly solved (Pfeuty 1970) and later shown to be equivalent to a two-dimensional classical Ising model in the extreme anisotropic limit (Suzuki 1976, Fradkin and Susskind 1978, Kogut 1979). The critical exponents are

$$\begin{aligned} &s = \nu = z = 1 \\ &\beta = \tfrac{1}{8}, \qquad \gamma = \varphi = \tfrac{7}{4}, \qquad \delta = 15, \qquad \eta = \tfrac{1}{4}. \end{aligned} \tag{2.32}$$

These results are consistent with

$$y_\varepsilon = 1, \qquad y_h = \tfrac{15}{8}. \tag{2.33}$$

The 'thermal' exponents of $\mathscr{H}_{m2}$ are the same as those of the ITF since the excitation spectrum and the ground-state energy per site are the same in both cases.

The magnetic exponents related to the x magnetisation may be deduced from equations (2.17) and (2.16) giving

$$\beta_x(m) = \tfrac{1}{8}m \tag{2.34}$$

and

$$\eta_x(m) = \tfrac{1}{4}m \tag{2.35}$$

for the decay exponent of the correlation at the critical point. These results lead to

$$y_{h_x}(m) = 2 - \tfrac{1}{8}m \tag{2.36}$$

and using the scaling laws one obtains

$$\gamma_x(m) = \varphi_x(m) = 2 - \tfrac{1}{4}m, \qquad \delta_x(m) = (16/m) - 1. \tag{2.37}$$

The XY exponents are recovered when $m = 2$

$$\beta_{XY} = \tfrac{1}{4}, \qquad \gamma_{XY} = \varphi_{XY} = \tfrac{3}{2}, \qquad \delta_{XY} = 7; \qquad \eta_{XY} = \tfrac{1}{2}. \tag{2.38}$$

These results agree with the known values (McCoy 1968) and with a recent calculation of the φ exponent (Müller and Shrock 1984). According to equation (2.13), the z magnetisation vanishes at $\lambda_c = 1$ with

$$\beta_z(m) = \tfrac{1}{4} \tag{2.39}$$

for any $m \geqslant 2$. The z magnetic exponents keep their XY values for any $m \geqslant 2$ and

$$y_{h_z}(m) = \tfrac{7}{4} \qquad m \geqslant 2. \tag{2.40}$$

3. Potts quantum chain with multisite interactions

3.1. Self-duality

On a linear chain with N sites and free ends, let us associate with each site j a Potts variable (Potts 1952) $n_j = 0, 1, \ldots q-1$ and two Potts operators $R_x(j)$ and $R_z(j)$ such that $|n_j\rangle$ is an eigenstate of $R_x(j)$ (Kogut 1980, Sólyom and Pfeuty 1981) and $R_z(j)$ acts like a flip operator on this state

$$\begin{aligned} &R_x(j)|n_j\rangle = \exp(\mathrm{i}2\pi n_j/q)|n_j\rangle \\ &R_z(j)|n_j\rangle = |n_j + 1\rangle \\ &|n_j + q\rangle \equiv |n_j\rangle. \end{aligned} \tag{3.1}$$

It will be convenient to use the Hermitian conjugates $R_x^\dagger(j) = R_x^{q-1}(j)$ and $R_z^\dagger(j) = R_z^{q-1}(j)$ such that

$$\begin{aligned} &R_x^\dagger(j)|n_j\rangle = \exp(-\mathrm{i}2\pi n_j/q)|n_j\rangle \\ &R_z^\dagger(j)|n_j\rangle = |n_j - 1\rangle. \end{aligned} \tag{3.2}$$

The following relations are easily deduced from the above given definitions

$$\begin{aligned} &R_x^q(j) = R_z^q(j) = 1 \\ &R_x(j)R_x^+(j) = R_z(j)R_z^+(j) = 1 \\ &R_z(j)R_x(j) = \exp(-\mathrm{i}2\pi/q)R_x(j)R_z(j) \\ &R_z^+(j)R_x(j) = \exp(\mathrm{i}2\pi/q)R_x(j)R_z^+(j). \end{aligned} \tag{3.3}$$

The Hamiltonian of the Potts quantum chain with multisite interactions is defined as follows

$$\mathcal{H}_{mn}(\lambda) = -\frac{1}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}\left(\prod_{k=0}^{m-1} R_z^p(j+k) + \mathrm{HC}\right) - \frac{\lambda}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}\left(\prod_{k=0}^{n-1} R_x^p(j+k) + \mathrm{HC}\right). \tag{3.4}$$

The first term leads to simultaneous flips of the R_x eigenstates on m successive sites along the chain whereas the second one is a quasi-Potts interaction (Enting 1975) on n successive sites between the R_x eigenstates $\delta_q(\Sigma_{k=0}^{n-1} n_{j+k})$. This results from the identity

$$\delta_q(r) = \frac{1}{q}\sum_{p=0}^{q-1}\exp\left(\mathrm{i}\frac{2\pi pr}{q}\right). \tag{3.5}$$

When $q = 2$, one recovers the spin-$\frac{1}{2}$ Hamiltonian with multisite interactions (equation (2.1)).

The dual operators

$$\begin{aligned} S_x(j) &= \prod_{k\geqslant 0} R_z(j-kn)R_z^+(j-kn-m) \\ S_z^+(j) &= \prod_{k\geqslant 0} R_x(j+km)R_x^+(j+km+n) \end{aligned} \tag{3.6}$$

satisfy the Potts algebra (equation (3.3)) and transform the Hamiltonian into its self-dual form

$$\begin{aligned} \mathcal{H}_{mn}(\lambda) &= -\frac{\lambda}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}\left(\prod_{k=0}^{m-1} S_z^p(j+k) + \mathrm{HC}\right) - \frac{1}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}\left(\prod_{k=0}^{n-1} S_x^p(j+k) + \mathrm{HC}\right) \\ &= \lambda\mathcal{H}_{mn}(\lambda^{-1}). \end{aligned} \tag{3.7}$$

As a consequence the system is critical when $\lambda_c^2 = 1$ if the transition is unique.

The class of Hamiltonians with m arbitrary and $n = 2$ has interesting properties like in the case of spin-$\frac{1}{2}$. One may then replace the quasi-Potts interaction in the last term by the usual Potts interaction $\delta_{n_j,n_{j+1}}$ (Potts 1952)

$$\mathcal{H}_{m2}(\lambda) = -\frac{1}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}\left(\prod_{k=0}^{m-1} R_z^p(j+k) + \mathrm{HC}\right) - \frac{\lambda}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}(R_x^p(j)R_x^{+p}(j+1) + \mathrm{HC}). \tag{3.8}$$

The self-duality is obtained by using the dual operators

$$\begin{aligned} S_x(j) &= \prod_{k\geqslant 0}\prod_{l=0}^{m-1} R_z(j-k+l) \\ S_z^+(j) &= \prod_{k\geqslant 0} R_x^+(j-km+1)R_x^2(j-km)R_x^+(j-km-1). \end{aligned} \tag{3.9}$$

When $m = 1$ we recover the QTF which is equivalent to the two-dimensional classical

Potts model in the extreme anisotropic limit. The model with $m=2$ is the analogue of the anisotropic XY chain.

3.2. Perturbation expansion of the spontaneous magnetisation when $n=2$

The Hamiltonian $\mathcal{H}_{m2}(\lambda)$ may be rewritten under a form which is appropriate for a λ^{-1} expansion of the ground-state energy per site

$$\mathcal{H}_{m2}(\lambda)/\lambda = \mathcal{H}_0 + V + N/q \tag{3.10}$$

where

$$\begin{aligned} \mathcal{H}_0 &= -\sum_{j=1}^{N} \delta_{n_j,n_{j+1}} - h_x \sum_{j=1}^{N} (q\delta_{n_j,0}-1) \\ V &= -\frac{1}{\lambda q} \sum_{j=1}^{N} \sum_{p=1}^{q-1} \prod_{k=0}^{m-1} R_z^p(j+k). \end{aligned} \tag{3.11}$$

$\mathcal{H}_0$ is diagonal in the basis of the eigenstates $|n_j\rangle$ of the R_x operator and includes the interaction with the external field h_x applied 'in the x direction' and favouring the state $n_j=0$. The perturbation V acts as a flip operator on this basis.

The x spontaneous magnetisation is given by

$$\langle m_x\rangle_m = \frac{1}{2(q-1)} \sum_{p=1}^{q-1} (\langle R_x^p(j)\rangle_m + \text{HC}) = \frac{q\langle\delta_{n_j,0}\rangle_m - 1}{q-1} = -\frac{1}{q-1}\frac{\partial e_0}{\partial h_x}\bigg|_{h_x=0} \tag{3.12}$$

where $e_0(h_x)$ is the field-dependent ground-state energy per site. Using standard methods (Kogut 1979, 1980), we obtain

$$\begin{aligned} \langle m_x\rangle_m = 1 &- \frac{m\lambda^{-2}}{4q} - \frac{(q-2)m\lambda^{-3}}{4q^2} - \frac{1}{q^3}\left(\frac{5(q-2)m}{9} + \frac{3m(m-1)(q-2)}{32}\right. \\ &\left. + \frac{15m+m^2-(q-1)m(2m+7)+12m(q-2)^2}{16}\right)\lambda^{-4} + \mathrm{O}(\lambda^{-5}). \end{aligned} \tag{3.13}$$

The spin-$\frac{1}{2}$ expansion is recovered when $q=2$. With $m=1$, one gets the PTF x magnetisation

$$\begin{aligned} \langle m_x\rangle_{\text{PTF}} = 1 &- \frac{\lambda^{-2}}{4q} - \frac{(q-2)\lambda^{-3}}{4q^2} - \frac{1}{q^3}\left(\frac{5(q-2)}{9} + \frac{16-9(q-1)+12(q-2)^2}{16}\right)\lambda^{-4} \\ &+ \mathrm{O}(\lambda^{-5}) \end{aligned} \tag{3.14}$$

and, to this order, the relation

$$\langle m_x\rangle_m = (\langle m_x\rangle_{\text{PTF}})^m \tag{3.15}$$

is verified, like in the spin-$\frac{1}{2}$ case and one may expect a decoupling into m independent PTF as above.

3.3. Decoupling of the Potts quantum chain with multisite interactions when $n=2$

The decoupling of the Potts quantum chain with m-site interactions into m independent Potts chains in a transverse field may be performed through a duality transformation.

The appropriate dual operators are still the self-dual operators of the chain in a transverse field (equation (3.9) with $m=1$):

$$\begin{aligned} S_x(j) &= \prod_{k\geqslant 0} R_z(j-k) \\ S_z^+(j) &= R_x(j)R_x^+(j+1) \end{aligned} \tag{3.16}$$

and the transformed Hamiltonian is

$$\mathscr{H}_{m2}(\lambda) = -\frac{\lambda}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}(S_z^p(j)+\text{HC}) - \frac{1}{2q}\sum_{j=1}^{N}\sum_{p=1}^{q-1}(S_x^p(j)S_x^{+p}(j+m)+\text{HC}). \tag{3.17}$$

The self-duality transformation was actually obtained by first decoupling the system into m independent chains. The known self-dual operators of the PTF were then used to obtain equation (3.9) via the inverse transformation.

According to equation (3.17) the zero-field ground-state energy and the excitation spectrum of the multisite model with $n=2$ are the same as for the PTF. The z-spontaneous magnetisation:

$$\langle m_z\rangle_m = \frac{1}{2(q-1)}\sum_{p=1}^{q-1}(\langle R_z^p\rangle_m + \text{HC}) \tag{3.18}$$

is transformed into

$$\begin{aligned} \langle m_z\rangle_m &= \frac{1}{2(q-1)}\sum_{p=1}^{q-1}(\langle S_x^{+p}(j-1)S_x^p(j)\rangle + \text{HC}) \\ &= \frac{1}{2(q-1)}\sum_{p=1}^{q-1}(\langle S_x^{+p}(j-1)\rangle_{\text{PTF}}\langle S_x^p(j)\rangle_{\text{PTF}} + \text{HC}) \end{aligned} \tag{3.19}$$

when $m\geqslant 2$. $\langle\ldots\rangle_{\text{PTF}}$ is a ground-state expectation value on one of the m independent chains. A new duality transformation

$$\begin{aligned} T_x(j_\alpha) &= \prod_{k\geqslant 0} S_z(j_\alpha - k) \\ T_z^+(j_\alpha) &= S_x(j_\alpha)S_x^+(j_\alpha+1) \end{aligned} \tag{3.20}$$

where $\alpha = 1$, m refers to one of the PTF, which leads to

$$\Delta_p = \langle S_z^p(j_\alpha)\rangle_{\text{PTF}} = \left\langle \prod_{k\geqslant 0} T_z^{+p}(j_\alpha+k)\right\rangle_{\text{PTF}}. \tag{3.21}$$

In the last term $\Pi_{k\geqslant 0}\, T_z^{+p}(j_\alpha+k)$ creates a kink at j_α between the ground state and one of the $(q-1)$ remaining degenerate ground states of the chain α. The Potts permutation symmetry between these states ensures that $\Delta_p = \Delta$ is independent of p. It follows that

$$\langle m_z\rangle_m = \Delta^2 = \langle m_x\rangle_{\text{PTF}}^2 \tag{3.22}$$

since the PTF magnetisation is given by

$$\langle m_x\rangle_{\text{PTF}} = \frac{1}{2(q-1)}\sum_{p=1}^{q-1}(\langle S_x^p(j_\alpha)\rangle_{\text{PTF}} + \text{HC}) = \Delta. \tag{3.23}$$

In order to get the x spontaneous magnetisation of the multisite model $\langle m_x\rangle_m$ defined in equation (3.12), we study the order parameter correlation function (Wu 1982)

$$\rho_{xx}(R)=\frac{1}{2(q-1)}\sum_{p=1}^{q-1}(\langle R_x^p(j)R_x^{\dagger p}(j+R)\rangle_m+\text{HC}) \tag{3.24}$$

which may also be written

$$\rho_{xx}(R)=\frac{qP(R)-1}{q-1} \tag{3.25}$$

where $P(R)=\langle\delta_{n_j,n_{j+R}}\rangle_m$ is the probability to find the Potts states at j and $j+R$ in the same eigenstate of R_x. The duality transformation (equation (3.16)) gives

$$\rho_{xx}(R)=\frac{1}{2(q-1)}\sum_{p=1}^{q-1}\left(\prod_{\alpha=1}^{m}\left\langle\prod_{k=0}^{R_\alpha}S_z^{\dagger p}(j_\alpha+k)\right\rangle_\alpha+\text{HC}\right) \tag{3.26}$$

where the expectation value is taken on the ground state of the chain α and R_α is of order R/m. The operator in the bracket creates a domain wall of length R_α between the ground state of the chain α and one of the $q-1$ other degenerate ground states. Due to the permutation symmetry between these states, its expectation value

$$\Gamma_p(R_\alpha)=\left\langle\prod_{k=0}^{R_\alpha}S_z^{\dagger p}(j_\alpha+k)\right\rangle_\alpha \tag{3.27}$$

is independent of the index p. Let $\Gamma(R_\alpha)$ be the common value, then

$$\rho_{xx}(R)=\prod_{\alpha=1}^{m}\Gamma(R_\alpha). \tag{3.28}$$

On the chain α, the PTF order parameter correlation function is

$$\rho_{xx}^{\text{PTF}}(R_\alpha)=\frac{1}{2(q-1)}\sum_{p=1}^{q-1}(\langle S_x^p(j_\alpha)S_x^{\dagger p}(j_\alpha+R_\alpha)\rangle_\alpha+\text{HC}) \tag{3.29}$$

and the self-duality transformation of equation (3.20) leads to

$$\rho_{xx}^{\text{PTF}}(R_\alpha)=\frac{1}{2(q-1)}\sum_{p=1}^{q-1}\left(\left\langle\prod_{k=0}^{R_\alpha}T_z^{\dagger p}(j_\alpha+k)\right\rangle_\alpha+\text{HC}\right)=\Gamma(R_\alpha) \tag{3.30}$$

which, together with equation (3.28), gives

$$\rho_{xx}(R)=[\rho_{xx}^{\text{PTF}}(R/m)]^m. \tag{3.31}$$

In the ordered state $\lambda\geqslant 1$ and in the limit $R\to\infty$, the order parameter correlation function gives the square of the magnetisation and

$$\langle m_x\rangle_m=\langle m_x\rangle_{\text{PTF}}^m \tag{3.32}$$

in agreement with the result of the perturbation expansion.

3.4. Critical exponents of the Potts quantum chain with multisite interactions when $n=2$

The Potts model in a transverse field is equivalent to the two-dimensional classical Potts model in the extreme anisotropic limit for which the thermal and magnetic exponents are known (den Nijs 1979, Black and Emery 1981, Nienhuis *et al* 1980,

Pearson 1980, Wu 1982)

$$\begin{aligned} &y_e = 3(1-u)/(2-u) \qquad q \leqslant 4 \\ &y_h = (3-u)(5-u)/4(2-u) \\ &0 \leqslant u \equiv (2/\pi)\cos^{-1}(\sqrt{q}/2) \leqslant 1. \end{aligned} \tag{3.33}$$

Using the results of § 2.4 one gets the PTF exponents

$$s = \nu = (2-u)/3(1-u), \qquad z = 1 \tag{3.34}$$

$$\begin{aligned} &\beta = \frac{1+u}{12}, \qquad \gamma = \frac{7-4u+u^2}{6(1-u)}, \qquad \varphi = \frac{7-4u+u^2}{2(2-u)}; \\ &\delta = \frac{(3-u)(5-u)}{1-u^2}, \qquad \eta = \frac{1-u^2}{2(2-u)}. \end{aligned} \tag{3.35}$$

Since the quantum Potts model with m-site interactions has the same ground-state energy and excitation spectrum as the PTF, its thermal exponents are also given by equation (3.34). The magnetic exponents follow from the results of § 2. Equation (3.32) gives

$$\beta_x(m) = m(1+u)/12 \tag{3.36}$$

and using equation (3.31) at the critical point, one gets the decay exponent

$$\eta_x(m) = m(1-u^2)/2(2-u). \tag{3.37}$$

These values are consistent with

$$y_{h_x}(m) = 2 - m(1-u^2)/2(2-u). \tag{3.38}$$

The scaling laws give

$$\begin{aligned} &\gamma_x(m) = \frac{4(2-u)-m(1-u^2)}{6(1-u)}, \qquad \varphi_x(m) = 2 - \frac{m(1-u^2)}{2(2-u)}, \\ &\delta_x(m) = \frac{8(2-u)}{m(1-u^2)} - 1. \end{aligned} \tag{3.39}$$

The z exponents, which are m-independent, may be deduced from equation (3.22)

$$\beta_z(m) = \frac{1+u}{6}, \qquad y_{h_z} = 2 - \frac{1-u^2}{2(2-u)} \qquad m \geqslant 2. \tag{3.40}$$

The critical exponents of the analogue of the XY model ($m = 2$) are given in table 1 for $0 \leqslant q \leqslant 4$.

Table 1. Critical exponents of the quantum Potts analogue of the anisotropic XY chain.

q	u	y_e	y_h	β	γ	φ	δ	η	ν	z
0	1	0	2	$\frac{1}{3}$	∞	2	∞	0	∞	1
1	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{43}{24}$	$\frac{5}{18}$	$\frac{19}{9}$	$\frac{19}{12}$	$\frac{43}{5}$	$\frac{5}{12}$	$\frac{5}{3}$	1
2	$\frac{1}{2}$	1	$\frac{7}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	7	$\frac{1}{2}$	1	1
3	$\frac{1}{3}$	$\frac{6}{5}$	$\frac{26}{15}$	$\frac{2}{9}$	$\frac{11}{9}$	$\frac{22}{15}$	$\frac{13}{2}$	$\frac{8}{15}$	$\frac{5}{6}$	1
4	0	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{1}{6}$	1	$\frac{3}{2}$	7	$\frac{1}{2}$	$\frac{2}{3}$	1

4. Discussion

Let us briefly discuss some possible extension of the present work. The Potts Hamiltonian $\mathcal{H}_{m2}$ has been written with a Potts interaction for the x component and a quasi-Potts interaction for the z component with m-site interactions. Although the Potts interaction is more complicated in a multisite term, it would be preferable to have the same type of interactions for both components. It is not clear whether the methods used here will remain applicable in this situation.

Until now there are no standard notations for the Potts operators. The ones used in this work were introduced in order to make clear the analogy with the Pauli spin-$\frac{1}{2}$ operators. They also suggest the introduction of a third operator R_y to complete this analogy. Using the complete set of Potts operators, one may study quantum Potts models of the Heisenberg type. Work in this direction is in progress.

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